UDC 539.375

STATE OF STRESS AND STRAIN NEAR THE END OF A GROWING CRACK IN AN ELASTIC-PLASTIC MEDIUM

PMM Vol. 40, № 2, 1976, pp. 337-345

V. A. IBRAGIMOV
(Ivano-Frankovsk)

(Received December 19, 1974)

The problem of determining the stress and strain fields in the neighborhood of the end of a crack in an elastic-plastic medium is examined. On the basis of an analysis of the energy balance, the inadequacy of solutions obtained in [1 — 3] and taking into account the effect of unloading along the crack edges, is shown for a hyperfine structure. It is found that one of the possible reasons for the origination of the mentioned situation is the use of deformation theory with isotropic hardening as well as the isotropic theory of flow for an ideal elastic-plastic body for this problem. The asymptotic solution based on the anisotropic theory of translation hardening is analyzed, a deduction is made about conservation of a singularity of the form $1 / r$ (for the case of crack propagation) for the product of the principal terms of the stress and strain fields.

1. We consider an elastic-plastic medium with a crack loaded by external forces at a sufficiently large distance from the tip. Let the crack be propagated quasistatically (inertia terms are not taken into account).

The problem of determining the stress and strain fields was analyzed by a number of authors, where all the complete and incomplete (asymptotic in the sense of expansions near the singularity) solutions have been obtained within the framework of two approaches. One assumes extrapolation of the solution for a fixed crack to the case of its propagation and has the deduction of constancy of the index of the singularity for the product of the main terms of the stress and strain tensors for the fundamental result when the physical nonlinearity increases. The product mentioned is hence on the order of $1 / r$ (see [4, 5], etc.). Solutions of this kind have been used, in particular, to formulate criteria of ultimate crack equilibrium [4].

Another, physically more consistent, approach has the aim of taking account of the unloading effect originating at points of the media adjacent to the crack edges and, perhaps, secondary additional loading. Available solutions result in a deduction about the reduction of the order of the singularities in the stresses and strains for elastic-plastic bodies [1 — 3]. These solutions are used below to analyze the energy balance near the crack tip.

Let us limit ourselves to the case of antiplane strain along the x_3-axis parallel to the crack edge, assuming that a rectilinear crack is propagated along the x_1-axis. Let the material be subject to deformation theory with linear hardening

$$\tau_i = \frac{\tau}{\gamma}\gamma_i; \quad \tau = G\gamma, \quad \gamma \leqslant \gamma_0; \quad \tau = \tau_0 + g(\gamma - \gamma_0), \quad \gamma > \gamma_0 \qquad (1.1)$$
$$d\gamma > 0; \quad 0 < g \leqslant G, \quad \tau_0 = G\gamma_0 \quad (i = 1, 2, \quad \tau^2 = \tau_i\tau_i, \quad \gamma^2 = \gamma_i\gamma_i)$$

For unloading we have

$$\tau_i - \tau_i^* = G(\gamma_i - \gamma_i^*), \quad \tau_i^* = \frac{\tau_*}{\gamma_*}\gamma_i^*, \quad \tau_* = \tau_0 + g(\gamma_* - \gamma_0) \tag{1.2}$$

and for secondary loading

$$\tau_i - \tau_i^{**} = \frac{\tau}{\gamma}(\gamma_i - \gamma_i^{**}), \quad \tau = \tau_{**} + g(\gamma - \gamma_{**}) \tag{1.3}$$

The quantities τ_i, γ_i in the relationships (1.1)—(1.3) are values of the shear stresses and strains and are marked with either one or two asterisks if the values mentioned precede the unloading or secondary loading; the subscript of the x_3-axis is omitted for brevity.

Taking account of the stationarity condition ($\delta / \delta x_1 + \delta / \delta l = 0$), it is easy to show (by following [4]) that the first law of thermodynamics for the considered class of singularities of the required functions is converted into

$$R \sum_{n=1}^{3} \int_{\Gamma_n} [(E_n + K_n)\cos\vartheta - A_n]\, d\vartheta = 2\gamma_s \tag{1.4}$$

$$dE = \tau_i d\gamma_i, \quad A = \tau_i \nu_i \left(\frac{\partial w}{\partial x_1}\right), \quad K = \frac{1}{2}\rho l^{\cdot 2}\left(\frac{\partial w}{\partial x_1}\right)^2$$

Arcs of the circles Γ ($x_i x_i = R^2$) lying in the primary loading, unloading, and additional regions are denoted respectively by Γ_n ($n = 1, 2, 3$); ϑ is the polar angle and w is the displacement along the x_3-axis. The left side of (1.4) is invariant relative to the selection of R (under the condition of smallness of R as compared to the characteristic body size and the crack length l) and governs the increment in the total energy when l is increased by a unity of length [4]. We note that invariance can also be proved directly by using the Green's theorem for each of the regions and taking account of the continuity of τ_i, γ_i.

We evaluate the strain energy for (1.1) – (1.3) and express it in terms of the stress deviator intensity. Limiting ourselves to the neighborhood D in which $\tau \gg \tau_0$, $\tau_* \gg \tau_0$, $\tau_{**} \gg \tau_0$, we obtain

$$2GE_1 = \tau^2, \quad 2GE_2 = \tau^2 + \tau_*^2(\lambda - 1) \tag{1.5}$$
$$2GE_3 = \tau^2\lambda + (\tau_*^2 - \tau_{**}^2)(\lambda - 1), \quad \lambda = G / g$$

The asymptotic solution of the problem formulated above has the structure [2]

$$\tau_2 + i\tau_1 = \frac{A_i}{z^m} + \frac{B_i}{x_2^m}, \quad \vartheta_{i-1} \leqslant \vartheta \leqslant \vartheta_i, \quad i = 1, 2, 3 \tag{1.6}$$

$$\vartheta_0 = 0, \quad \vartheta_3 = \pi, \quad B_1 = \operatorname{Im} A_1 = 0$$

where A_i, B_i are some complex constants sought to the accuracy of an undetermined multiplier A_1 from the condition of continuity on the boundaries $\vartheta = \vartheta_1$, ϑ_2 of the domains. The index m of the singularity is determined by the equation

$$\lambda \operatorname{tg} m\vartheta_1 \operatorname{tg} m(\pi - \vartheta_1) = 1 \tag{1.7}$$

and satisfies the inequality $0 < m \leqslant {}^1/_2$ for $0 < g \leqslant G$.

Inserting (1.5) and (1.6) into (1.4), we have

$$R^{1-2m}(A_1^2, \vartheta_1, \vartheta_2, \lambda, m) = 2\gamma_s \tag{1.8}$$

Equation (1.8) cannot be satisfied for arbitrary R with $m < 1/2$ (the purely elastic case $g = G$ is thereby excluded), which results in a conclusion about the incorrectness, in the sense mentioned, of solutions of the type (1.6). An analogous deduction is also valid for problems within the framework of an ideally plastic model [1, 3]. It is easy to see this latter by an analogous means, by using the modification of (1.4) obtained in [4] (see formula (3.5)).

On the basis of the above, a deduction can be made about the fact that under the condition $\gamma_s \neq 0$ the singularity of the product of the principal stress and strain tensor terms should be on the order of $1 / r$ in a small neighborhood of the crack tip, as in the case of a fixed crack.

2. Let us consider some fundamental assumptions underlying the solution of the problem posed in Sect. 1, from the viewpoint of their influence on the nature of the singularity of the solutions. Among them are: the quasistatic nature of the solution, its stationarity, and the use of isotropic theories of plasticity.

1°. We examine the asymptotic solution of the problem on the dynamic stationary propagation of the semi-infinite crack. Introducing a moving coordinate system x_i placed at the crack tip, let us assume (ξ_i is the fixed system)

$$\xi_1 = x_1 + ct, \quad \xi_2 = x_2, \quad \xi_3 = x_3 \tag{2.1}$$

Using the equations of motion and strain compatibility $\tau_{1,1} + \tau_{2,2} = \rho w^{\cdot\cdot}, \gamma_{1,2} - \gamma_{2,1} = 0$ in the fixed system, as well as the relationships (1.1) — (1.3), we obtain in the x_i-axes

$$\begin{aligned}
&\alpha w_{,11} + w_{,22} = 0, \quad 0 \leqslant \vartheta \leqslant \vartheta_1 \qquad (2.2)\\
&\beta w_{,11} + w_{,22} = \mu \overset{*}{\gamma}_{2,2}, \quad \vartheta_1 \leqslant \vartheta \leqslant \vartheta_2\\
&\alpha w_{,11} + w_{,22} = \mu (1 - \mu)^{-1} (\overset{*}{\gamma}_{2,2} - \overset{*}{\gamma}_{2,2}), \quad \vartheta_2 \leqslant \vartheta \leqslant \pi\\
&\alpha = 1 - \rho c^2 / g, \quad \beta = 1 - \rho c^2 / G, \quad \mu = 1 - g / G
\end{aligned}$$

The boundary conditions are:

$$w_{,1} = 0 \ (\vartheta = 0), \quad [w_{,1}] = [w_{,2}] = [w] = 0 \quad (\vartheta = \vartheta_1, \vartheta_2) \tag{2.3}$$

(the square brackets denote jumps in the function). Moreover, $w_{,2} = \mu\gamma_2^*(\vartheta = \pi)$ in the absence of a secondary loading zone and $w_{,2} = \mu (1 - \mu)^{-1} (\gamma_2^* - \gamma_2^{**})$ when it originates. Keeping in mind the clarification of the characteristic singularities of the solution, let us limit ourselves to the first of the cases mentioned, for brevity. We first consider that $\alpha > 0$, $\beta > 0$ $(\alpha \leqslant \beta)$, which corresponds to values of the velocity $c < (g / \rho)^{1/2}$.

We seek the solution of the first equation in (2.2) as

$$w = Cr^n \sin n\varphi \quad (\sqrt{\alpha}\, \mathrm{tg}\vartheta = \mathrm{tg}\, \varphi, \ r^2 = x_1^2 + \alpha x_2^2) \tag{2.4}$$

from which

$$\gamma_2^* = C (m + 1)\alpha^{(m+1)/2} x_2^m (\sin \varphi_1)^{-m} \cos m\varphi_1, \quad m = n - 1$$
$$\mathrm{tg}\, \varphi_1 = \sqrt{\alpha}\, \mathrm{tg}\, \vartheta_1$$

Using the second of Eqs. (2.2), we obtain in the unloading zone

$$w = s^n (A \cos n\omega + B \sin n\omega) + \mu \int_0^{\sqrt{\alpha} x_2} \gamma_2^* dx_2 \tag{2.5}$$

$$(\sqrt{\beta}\, \mathrm{tg}\, \vartheta = \mathrm{tg}\, \omega, \ s^2 = \alpha (x_1^2 + \beta x_2^2))$$

Computing the strain in (2.4) and (2.5) and using together with (2.3) the condition on the free boundary $\vartheta = \pi$, we obtain a system of equations to find the constants A, B, C. Nontrivial solutions of this system exist if

$$\sqrt{\alpha}\,(1-\mu)\,\cos m\varphi_1 \cos m\,(\omega_1-\pi) + \sqrt{\beta}\,\sin m\varphi_1 \sin m\,(\omega_1-\pi) = 0 \quad (2.6)$$
$$(\mathrm{tg}\ \omega_1 = \sqrt{\beta}\ \mathrm{tg}\,\vartheta_1)$$

In the static case $(\alpha = \beta = 1)$ this equation turns into (1.7). Using (2.6) it is easy to see that the value $m = -{}^1/_2$ which assures the required order of the singularity $\tau_i\gamma_i$ as $r \to 0$ is possible, as in the static case, for a linearly elastic medium $(\mu = 0)$ only.

For $(g / \rho)^{1/2} < C < (G / \rho)^{1/2}$ the equation for the displacement in the first zone is of hyperbolic type. The solution has the form

$$w = C\,[(x_1 + \sqrt{\alpha_1}x_2)^n - (x_1 - \sqrt{\alpha_1}x_2)^n], \quad \sqrt{\alpha_1}x_2 \leqslant x_1, \quad \alpha_1 = \rho c^2 / g - 1$$

The representation (2.5) is conserved in the second zone with the appropriate replacement for γ_2^* in the second member. Omitting the transformations, we present the analog to the characteristic equation (2.6):

$$\sqrt{\beta}\,\sin m\,(\pi-\omega_1)\ [(\cos\varphi_1 + \sin\varphi_1)^m + (\cos\varphi_1 - \sin\varphi_1)^m] - \sqrt{\alpha_1}\,(1-\mu)\,[(\cos\varphi_1 + \sin\varphi_1)^m - (\cos\varphi_1 - \sin\varphi_1)^m]\cos m\,(\pi-\omega_1) = 0 \quad (2.7)$$

Analysis of (2.7) shows that no solution of the required order exists even for the considered interval of crack propagation velocity values. We note that the case $\alpha < 0$, $\beta < 0$ is physically impossible.

Therefore, taking account of the inertia terms does not alter the deduction made in Sect. 1, about the inadequacy of solutions of the type (1.6) and (2.5), since all the members in the left side of (1.4), including the kinetic enegry, are of the order of R^{1+2m}.

2°. To estimate the influence of the assumption about the existence of a stationary crack propagation mode in an elastic-plastic material, it is sufficient to consider the following formulation of the problem.

Let the advancement of the crack be quasistatic and nonstationary so that all the required fields and the parameters $m, \vartheta_1, \vartheta_2$ depend explicitly on l. We consider the state of the medium preceding the beginning of crack propagation and corresponding to some value of the stress intensity factor K_0

$$\tau_2^\circ + i\tau_1^\circ = K_0 / (z - l_0)^{1/2} \quad (2.8)$$

(let us recall that (2.8) is an asymptotic solution $(\tau \gg \tau_0)$ of the problem of a fixed crack in a medium with the law (1.1)) and the adjacent state characterized by the value $K = K_0 + \Delta K$ $(\Delta K \leqslant K_0)$, and $\tau_i = \tau_i^\circ + \Delta\tau_i$, $w = w_0 + \Delta w$, $l = l_0 + \Delta l$.

Converting the first law of thermodynamics to the case under consideration, we obtain

$$\oint_\Gamma \tau_i^\circ \nu_i w' d\Gamma - \frac{1}{g}\int_{D_1} \tau_i^\circ \tau_i' dx\,dy - \frac{1}{G}\int_{D_2} \tau_i^\circ \tau_i' dx dy = 2\gamma_s \quad (2.9)$$

Here D_1, D_2 are loading and unloading zones (we keep in mind the absence of secondary loading, as before), $w' = \partial w / \partial l$.

The asymptotics

$$w' \sim r^{-1/2}, \quad \tau_i' \sim r^{-3/2}, \quad r \to 0 \quad (2.10)$$

follows from (2.8) and (2.9).

To seek the solution, we use the theory of flow with isotropic linear hardening

$$G\gamma_r' = \tau_r' + \mu\tau_0^{-1}\tau'\tau_r^\circ, \quad G\gamma_\vartheta' = \tau_\vartheta' + \mu\tau_0^{-1}\tau'\tau_\vartheta^\circ$$
$$\tau'\tau_0 = \tau_r^\circ\tau_r' + \tau_\vartheta^\circ\tau_\vartheta', \quad \mu = G/g - 1, \quad z = l_0 + re^{i\vartheta}$$
$$\tau_0^2 = \tau_i^\circ\tau_i^\circ = f(q), \quad dq = (2d\gamma_i^p d\gamma_i^p)^{1/2}, \quad f'(q) > 0$$

We should set $\mu = 0$ in the unloading domain.

The equilibrium equations are satisfied identically if the function $\Phi(r, \vartheta)$ ($\tau_r' = \partial\Phi / r\partial\vartheta, \tau_\vartheta' = -\partial\Phi / \partial r$) is introduced. Assuming $\Phi = r^m \Psi(\vartheta)$ under the condition $-1/2 \leqslant m < 0$, it is easy to show that there follows from the compatibility equation of the strain rates, the boundary conditions, and the continuity conditions for the stress and strain rates, it follows

$$\Psi''\left(1 + \mu\sin^2\frac{\vartheta}{2}\right) - \left(m - \frac{1}{2}\right)\mu\sin\vartheta\Psi' + \tag{2.11}$$
$$\Psi m\left[(m-1)\left(1 + \mu\cos^2\frac{\vartheta}{2}\right) + 1 + \frac{\mu}{2}\right] = 0, \quad 0 \leqslant \vartheta \leqslant \vartheta_1; \quad \Psi'(0) = 0$$

$$\Psi'' + m^2\Psi = 0, \quad \vartheta_1 \leqslant \vartheta \leqslant \pi; \quad \Psi(\pi) = 0; \; [\Psi(\vartheta_1)] = [\Psi'(\vartheta_1)] = 0 \tag{2.12}$$

Moreover, for $\vartheta = \vartheta_1$ the loading is neutral

$$m\Psi(\vartheta_1)\cos m(\pi - \vartheta_1) + \Psi'(\vartheta_1)\sin m(\pi - \vartheta_1) = 0 \tag{2.13}$$

As is easy to verify, the boundary value problem (2.12), (2.13) has a nontrivial solution if $\sin[m\pi + (1/2 - m)\vartheta_1] = 0$. Determing the value of the angle ϑ_1 as a function of the exponent of the singularity we have

$$\vartheta_1 = m\pi / (m - 1/2) \tag{2.14}$$

Now, let us examine the boundary value problem (2.11), (2.13), which should have the eigenvalue $m = -1/2$, because of (2.10), and show that it has no nontrivial solutions for $\mu \neq 0$.

Assuming that the solution is analytic in the neighborhood of $\mu = 0$

$$\Psi(\vartheta, \mu) = \Psi_0(\vartheta) + \mu\Psi_1(\vartheta) + \mu^2\Psi_2(\vartheta) + \ldots \tag{2.15}$$

it is clear that the function $\Psi_0(\vartheta)$ yields the solution of the linear elastic problem.

For the sequel it is sufficient to consider the case $\mu = o(1)$, hence, retaining terms not above the first order in the expansion (2.15), and inserting them into (2.11), we obtain equations to determine the functions $\Psi_0(\vartheta)$, $\Psi_1(\vartheta)$

$$L[\Psi_0(\vartheta)] = 0, \quad L[\Psi_1(\vartheta)] + M[\Psi_0(\vartheta)] = 0, \quad L[\Psi] \equiv \Psi'' + \frac{1}{4}\Psi \tag{2.16}$$
$$M[\Psi] = \Psi''\sin^2\frac{\vartheta}{2} + \Psi'\sin\vartheta - \frac{1}{4}\Psi\left(1 - 3\cos^2\frac{\vartheta}{2}\right)$$

Omitting the appropriate transformations, we indicate that the solutions of (2.16) are easily sought in explicit form and contain four arbitrary constants. These latter are determined from a system of linear equations resulting from the boundary conditions in (2.11), (2.13) and the evenness condition for $\Psi_0(\vartheta)$, $\Psi_1(\vartheta)$. The system mentioned has nontrivial solutions for $\mu = 0$ only.

Therefore, the asymptotic form (2.10) cannot be obtained for $\mu \neq 0$ and therefore the assumption about the stationarity of the crack development mode is not a reason for the inadequacy of the solutions under consideration.

3°. Let us consider the question of the validity of using different plasticity theories for problems of crack advancement.

We show first that deformation theory with isotropic hardening is inconsistent for problems of the type mentioned. In fact, loading at any point belonging to a small neighborhood of the tip differs from the proportional in contrast to the case of a fixed crack, hence, use of finite "stress-strain" relations can result in violation of the Drucker postulate. Budiansky [6] showed that the deformation theory is not contradictory in the mentioned sense if the relationship

$$\alpha < \beta \quad (\mathrm{tg}\,\beta = (G_s G - G_t G_s) / (G_t G - G_t G_s)) \tag{2.17}$$

is satisfied for any instant of the loading process, where α is the angle between the radius-vector of a point in the stress deviator space and the tangent to the loading trajectory, G_t, G_s are the tangent and secant moduli, and G is the unloading modulus.

In the plasticity theory (1.1) — (1.3) we have $G_t = G_s$ (for the asymptotic solution) and because of (2.17)

$$\beta = \pi / 4, \quad \alpha < \pi / 4 \tag{2.18}$$

Using the solution (1.6), for the loading zone, say, we find the boundary of the integral for the value of ϑ corresponding to the inequality (2.18). Varying (1.6), we find that

$$\delta\tau_{1,2} = \pm Amr^{-m-2}\left[\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}\frac{(m+1)\,\vartheta}{\cos\vartheta}\right]\delta x_1$$

Taking into account that $\tau\delta\tau\cos\alpha = \tau_i\delta\tau_i$, we have $\alpha = \vartheta$ for the loading angle α We hence obtain the required inequality $\vartheta < \pi / 4$ from (2.18). Hence, the inequality (2.17) is violated for the sector $\pi / 4 \leqslant \vartheta \leqslant \vartheta_1$ (belonging to the loading zone for $\vartheta_1 > \pi / 4$), and the use of the deformation theory cannot be considered justified. This latter is also clear in connection with the known fact of the absence of continuity of the plastic shear modulus during the transition from active loading to unloading [7]. Hence, none of the neutral loading conditions $\lim \tau' = 0$ is successfully satisfied for $\vartheta \to \vartheta_1 \pm 0$ in the solution of the boundary value problem (which results in the loss of a relationship analogous to (2.14)).

The problems considered in [1] as well as in Sect. 2°, yield a foundation, to a known degree, for pessimistic deductions relative to the use of isotropic flow theory also. It is known [7] that test data verify the theory of flow with anisotropic hardening more completely for multilinked loading trajectories. Hence, one of the versions of a theory of the kind mentioned is examined below.

3. Let us assume that hardening is translational in nature and there is no initial anisotropy of the medium. In the linear hardening case the governing relationships are

$$\gamma_i^{\cdot} = \gamma_i^{\cdot e} + \gamma_i^{\cdot p}, \quad \gamma_i^{\cdot e} = \frac{\tau_i^{\cdot}}{G}, \quad \gamma_i^{\cdot p} = \frac{1}{ak^2}(\tau_i^{\circ}\tau_i^{\cdot})\,\tau_i^{\circ} \tag{3.1}$$

$$\tau_i^{\circ} = \tau_i - s_i, \quad s_i = a\gamma_i^{p}, \quad \tau_i^{\circ}\tau_i^{\circ} = k^2, \quad a > 0, \quad i = 1, 2$$

Using the relationships (3.1), we estimate the residual strains in the neighborhood of a point of the medium at a short distance x_2 form Ox_1. The loading trajectory in the stress space is shown by the line *oabco* in Fig. 1 and is close to the four-section broken line $OABCO$. The initial loading surface S_0 is the circle of radius k (k is the pure shear yield point). Integrating (3.1) along $OABC$ and marking the values of the elastic and plastic strain components corresponding to the angular points of the trajectory by superscripts, we obtain

$$(OA):\quad \tau_1 = 0,\quad \tau_1{}^{\circ} = 0,\quad s_1 = 0,\quad \gamma_1{}^p = \gamma_1{}^e = 0$$
$$\tau_2{}^{\circ} = \tau_2 - s_2 = k,\qquad 0 \leqslant \tau_2 \leqslant T,\quad (\gamma_2{}^p)_A = (T - k)\ /\ a$$
$$(\gamma_2{}^p)_A = T\ /\ G \quad (T \to \infty)$$
$$(AB):\quad -T \leqslant \tau_1 \leqslant 0,\quad \tau_2 = T,\quad \tau_2{}^{\circ} = T - s_2,\quad (\gamma_1{}^p)_B =$$
$$(\varkappa - T)\ /\ a \quad (\gamma_1{}^e)_B = -T\ /\ G,\quad (\gamma_2{}^p)_B = (T + q)\ /\ a$$
$$(\gamma_2{}^e)_B = (T + q)\ /\ G$$
$$q^2 = k^2 - \varkappa^2,\quad \varkappa = k\,(1 - e^{-2T/k})\ /\ (1 + e^{-2T/k})$$
$$(BC):\quad \tau_1 = -T,\quad 0 \leqslant \tau_2 \leqslant T,\quad \tau_1{}^0 = -T - s_1,\qquad (\gamma_2{}^e)_c = 0$$
$$(\gamma_2{}^p)_c = \frac{k}{a}\left(1 - e^{-2T/k}\frac{k-q}{k+q}\right)\left(1 + e^{-2T/k}\frac{k-q}{k+q}\right)$$
$$(\gamma_1{}^e)_c = -\frac{T}{G},\quad (\gamma_1{}^p)_c = -\frac{T}{a} + \frac{1}{a}\,[k^2 - (a\gamma_2{}^p)_c{}^2]^{1/2}$$
$$(\gamma_1{}^p)_c \approx -\frac{T}{a} \quad (T \to \infty)$$

Keeping in mind the asymptotic analysis (the trajectory abc), let us give an estimate of the residual (inelastic) strains near the upper edge of the crack

$$\gamma_1{}^p = -\frac{T}{a} + O(1),\quad \gamma_2{}^p = O(1),\quad T \to \infty \tag{3.2}$$

Therefore, the shear strain $\gamma_1{}^p$ evaluated by the theory of translational hardening is large while the quantity $\gamma_2{}^p$ remains bounded. Within the framework of isotropic theory an analogous deduction is valid only for a fixed crack [4] (upon conservation of the singularity $1\ /\ r$). It should be noted that the governing relationships of the theory under consideration correspond to strain with negative dissipation for certain loading paths, hence, the explanation presented above is apparently not uniquely possible.

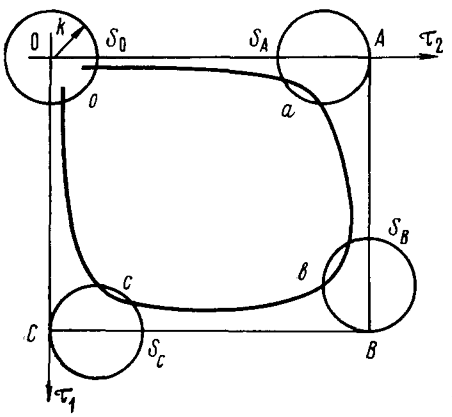

Fig. 1

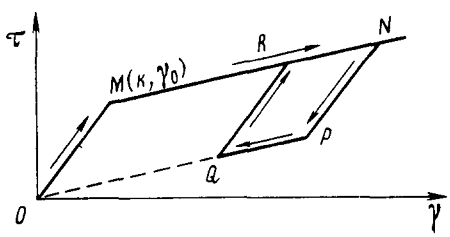

Fig. 2

4. Let us consider the asymptotic solution of the problem of quasistatic stationary crack growth in an elastic-plastic material with the linear translational hardening (3.1).

For simplification, we formulate final relationships of deformation theory corresponding to the relationships (3.1) under a uniaxial loading. As is easy to see, these latter are

$$\tau_i = G\gamma_i \quad (\gamma \leqslant \gamma_0) \tag{4.1}$$

$$\tau_i = \frac{\tau}{\gamma}\gamma_i, \quad \tau = k + g(\gamma - \gamma_0) \quad (\gamma \geqslant \gamma_0, \quad d\gamma > 0) \tag{4.2}$$

$$\tau_i - \tau_i^* = G(\gamma_i - \gamma_i^*) \quad (\gamma_{**} \leqslant \gamma \leqslant \gamma_*, \quad d\gamma < 0) \tag{4.3}$$

$$\gamma_{**} - \gamma_* - \frac{2k}{G}$$

$$\tau_i - \tau_i^{**} = \frac{\tau}{\gamma}(\gamma_i - \gamma_i^{**}) \quad (\gamma \leqslant \gamma_{**}, d\gamma < 0) \tag{4.4}$$

$$\tau = \tau_{**} + g(\gamma - \gamma_{**})$$

$$\tau_i - \tau_i^{***} = G(\gamma_i - \gamma_i^{***}) \quad (\gamma < \gamma_{**}, \quad d\gamma > 0), \quad \tau_{**} = \tau_* - 2k \tag{4.5}$$

The stress and strain components in (4. 1) — (4. 5) which are marked by one, two and three asterisks, are, respectively, related to the dependences (4. 2) — (4. 4), where we should substitute $\gamma = \gamma_*, \gamma_{**}, \gamma_{***}$.

For the uniaxial case the $\tau \sim \gamma$ diagram is shown in Fig. 2 ($N = N(\tau_*, \gamma_*)$, $P = P(\tau_{**}, \gamma_{**})$, $Q = Q(\tau_{***}, \gamma_{***})$), from which it follows that the loading at any point of the medium $|x_2| = o(l)$ in the primary loading zone in front of the crack tip follows the section MN. Unloading sets in during crack growth by first following the section NP with modulus G and then the section PQ (modulus g).

If there is a secondary loading zone, the state of stress follows the links QR, RN, etc. The solution is

$$\tau_2 + i\tau_1 = \frac{A_1}{z^m}, \quad 0 \leqslant \vartheta \leqslant \vartheta_1 \tag{4.6}$$

$$\tau_2 + i\tau_1 = \frac{A_\nu}{z^m} + \frac{B_\nu}{x_2^m}, \quad \vartheta_{\nu-1} \leqslant \vartheta \leqslant \vartheta_\nu$$

$$m > 0, \quad \operatorname{Im} A_1 = 0, \quad r \gg k; \quad \nu = 2, \ldots, n$$

where ϑ_1, ϑ_ν are the asymptotic slopes of the tangents to the boundaries of the primary loading zone, the lines separating the unloading zones ($\nu = 2, \ldots$) and the corresponding points $N, P, Q, R, \ldots$; m is the exponent of the singularity which is not known in advance, and n is the number of the zones.

Evaluating the intensity of the shear stresses in (4. 6) for $\vartheta_1 \leqslant \vartheta \leqslant \vartheta_2$ and taking account of the conditions

$$\tau(\vartheta_1) = \tau_*, \quad \tau(\vartheta_2) = \tau_{**}, \quad \tau_{**} - \tau_* = 2k$$

we obtain

$$x_2^{-m/2}[A_2(\sin\vartheta_1)^m e^{-im\vartheta_1} + B_2] - x_2^{-m/2}[A_2(\sin\vartheta_2)^m e^{-im\vartheta_2} + B_2] = 2k \tag{4.7}$$

Taking account of the singular nature of the solution and retaining terms of the order of $x_2^{-m/2}$ in (4. 7) as $x_2 \to 0$, we see that a nontrivial solution exists only for $\vartheta_1 = \vartheta_2$. This should have been expected since it is evident in advance that the zone $\vartheta_1 \leqslant \vartheta \leqslant \vartheta_2$ corresponding to NP is degenerate for the asymptotic solution ($r \to 0$, $\tau_* \to \infty$) at the line $\vartheta = \vartheta_1 = \vartheta_2$. We note that the assertion proved is equivalent to identification of the points N and P of the diagram.

An analogous deduction is evidently valid for any zone in which the relationships with modulus G from (4. 1) — (4. 5) hold (for example, for the section QR in Fig. 2). Therefore, for points of the media lying in a small neighborhood of the crack tip, the strain asymptotically follows the hardening section (4. 2).

Now, requiring continuity of the stress on $\vartheta = \vartheta_\nu$ and complying with the condition $\tau_2 = 0$ for $\vartheta = \pi$, omitting simple manipulations, we obtain $B_\nu = 0$, $A_\nu = A_1$, $m = {}^1/_2$ $(\nu = 2, \ldots, n)$. It follows that the singular solution of the problem has the same exponent of the singularity for the linear hardening case as the elastic solution has. Hence, the principal term is $\tau_i\gamma_i \sim 1 / r$ as $r \to 0$. This last result is easily extended to the nonlinear hardening case.

Within the framework of the concepts of quasi-brittle fracture, it also follows from the relationships (1.4) and (4.6) that the ultimate equilibrium criterion, given a rigorous foundation for linearly elastic media, is valid even under elastic-plastic strain (the quantity A_1 has the meaning of the coefficient of intensity of the hyperfine structure).

We note that the complete solution of the problem which takes account of the τ_i, γ_i fields far from the tip (and, in particular, governing the value of A_1) will differ from the linearly elastic value, hence, A_1 does not agree with the fine structure intensity coefficient. It follows from dimensional analysis that

$$A_1 = f\left(\frac{g}{G}, \frac{l}{L}, \frac{\tau_\infty}{G}\right) g l^{1/2}$$

(L is the characteristic dimension of the body, τ_∞ is the external load), hence, the quasi-brittle fracture criterion can depend on the ratio between the tangent modulus and the unloading modulus.

The author is grateful to L. A. Galin, V. D. Kliushnikov and G. P. Cherepanov for valuable discussions of the results of the research.

REFERENCES

1. Chitaley, A. and McClintock, F., Elastic-plastic mechanics of steady crack growth under anti-plane shear. J. Mech. and Phys. Solids, Vol. 19, № 3, 1971.

2. Slepian, L. I., Deformation at the edge of a growing crack. Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, № 4, 1973.

3. Slepian, L. I., A growing crack under plane strain of an elastic-plastic body. Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, № 1, 1974.

4. Cherepanov, G. P., Crack propagation in continuous media. PMM Vol. 31, № 3, 1967.

5. Rice, J. R. and Rosengren, G. F., Plane strain deformation near a crack tip in a power-law hardening material. J. Mech. and Phys. Solids, Vol. 16, № 1, 1968.

6. Budiansky, B., A reassessment of deformation theories of plasticity. J. Appl. Mech., Vol. 26, № 2, 1959.

7. Kachanov, L. M., Theory of Plasticity. "Nauka", Moscow, 1969.

Translated by M. D. F.